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Online publication date: 11 November 2010

LIQUID CRYSTALS, 1997, VOL. 23, NO. 4, 525–529

Fréedericksz transitions in smectic liquid crystals in annular geometries

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(*Received 24 January 1997; accepted 16 May 1997*)

This paper investigates the stability of simple static orientation patterns in a sample of smectic liquid crystal confined to a cylindrical annulus, when a magnetic field is applied. Four different arrangements are considered, covering cases where the layer normal is everywhere either radial or axial, and the (orthogonal) magnetic field is either radial, azimuthal or axial. A classification is given of the threshold radii for mechanical instabilities, and of the threshold magnetic fields for Fréedericksz transitions for these cases, with strong anchoring at the boundaries.

1. Introduction

As is well known, a Fréedericksz transition occurs in a liquid crystal when, under an incrementally increasing applied magnetic or electric field, a simple initial equilibrium orientation pattern in the material begins to change to a more distorted pattern, at some critical value of the applied field. Theoretical studies of Fréedericksz transitions in nematic materials have been very successful in providing means of verifying experimentally the continuum theory of Ericksen and Leslie [1] for these materials, and of measuring the elastic constants that occur in that theory. In particular there has over the past three decades been a variety of studies concerning orientation patterns in nematic samples confined to a cylindrical annulus. For example, the early analyses of Leslie [2] and Atkin and Barratt [3] concerned, respectively, the case in which the material is initially aligned azimuthally with the magnetic field applied in a radial direction, and the case in which the initial alignment is axial and the applied magnetic field is either radial or azimuthal. For these set-ups, on the assumption of strong anchoring at the boundaries, critical magnetic fields at which Fréedericksz transitions occur were determined in terms of the elastic constants k_{11}, k_{22} and k_{33}.

Much more recently Palffy-Muhoray *et al.* [4] considered the onset of purely mechanical instabilities in nematic samples in an annulus, their analysis indicating that, even with no applied field, an initial radial or azimuthal alignment would be unstable if the outer radius were to exceed some critical value r_c. In particular they showed that, when there is weak anchoring on at least one cylindrical boundary, r_c depends on the saddle–splay elastic constant k_{24} as well as various anchoring coefficients and other elastic constants. Barratt and Duffy [5] extended their analysis to include the effects of magnetic fields: they presented a catalogue of the threshold fields H_c required to induce a Fréedericksz transition for all the six cases in which the (mutually orthogonal) magnetic field and initial director field are either radial, azimuthal or axial.

The success of studies of nematics in annular geometries has motivated work on analogous problems for smectic C materials. Atkin and Stewart [6], using the continuum theory for smectics recently formulated by Leslie *et al.* [7], considered the case in which the smectic layers form circular cylinders concentric with the boundaries, with the initial director orientation everywhere lying in the radial–axial plane and with the magnetic field applied azimuthally. They presented a non-linear study based on static theory, and showed that a more distorted state becomes energetically more favourable than the initial alignment when the field exceeds a critical value H_c. The relationship between H_c and various material parameters provides a means of estimating certain elastic constants. Here we consider four arrangements in which the smectic layers are either concentric cylinders or circular discs, and the applied field is either radial, axial or azimuthal. (The number of possible arrangements of this type is less than for nematics due to the constraint that the angle between

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the director and the layer normal is fixed.) After outlining in §2 the continuum theory of Leslie *et al.* [7], we consider in §3 the linear stability of the initial alignments with respect to time-dependent perturbations in the director and velocity fields. We catalogue both the critical radii for mechanical instabilities and the threshold magnetic fields for Fréedericksz transitions in all four cases (and we show agreement with the threshold field obtained by Atkin and Stewart [6] for the case they considered).

The continuum theory of Leslie *et al.* [7] for smectics involves nine elastic constants, and our results involve five of these constants; thus potentially these five could be determined by means of experimental arrangements of the type considered herein.

2. The continuum theory

Here we briefly summarize the equations proposed by Leslie *et al.* [7] to describe the isothermal behaviour of incompressible smectic C liquid crystals. Their continuum model assumes that these materials are uniformly layered structures with the average molecular orientation (represented by the director $\mathbf{n}$) having a uniform tilt angle α with respect to the unit normal $\mathbf{a}$ to the layer. Two orthonormal vectors are employed to describe this layered structure; one is $\mathbf{a}$ and the other is the unit orthogonal projection $\mathbf{c}$ of the director onto the smectic planes, so that

$$\mathbf{n}=\mathbf{a}\cos\alpha+\mathbf{c}\sin\alpha. \tag{1}$$

The relevant equations are then the constraints

$$\begin{aligned}\mathbf{a}\ \ \mathbf{a}=\mathbf{c}\ \ \mathbf{c}&=1,\\ \mathbf{a}\ \ \mathbf{c}&=0,\\ \operatorname{curl}\mathbf{a}&=\mathbf{0},\\ \nabla\ \ \mathbf{v}&=0,\end{aligned} \tag{2}$$

and, in Cartesian tensor notation, the balance laws

$$\rho\dot{v}_i=-\tilde{p}_{,i}+\tilde{g}^a_j a_{j,i}+\tilde{g}^c_j c_{j,i}+\tilde{t}_{ij,j}, \tag{3}$$

$$\left(\frac{\partial W}{\partial a_{i,j}}\right)_{,j}-\frac{\partial W}{\partial a_i}+\tilde{g}^a_i+G^a_i+\varepsilon_{ijk}\beta_{k,j}+\gamma a_i+\kappa c_i=0 \tag{4}$$

and

$$\left(\frac{\partial W}{\partial c_{i,j}}\right)_{,j}-\frac{\partial W}{\partial c_i}+\tilde{g}^c_i+G^c_i+\kappa a_i+\tau c_i=0, \tag{5}$$

where

$$\left.\begin{aligned}
&\tilde{p}=-H_{\mathrm{m}}+p+W,\qquad \tilde{t}_{ij}=\tilde{t}^{\mathrm{s}}_{ij}+\tilde{t}^{\mathrm{ss}}_{ij},\\
&\tilde{t}^{\mathrm{s}}_{ij}=\mu_0 D_{ij}+\mu_1 a_p D^a_p a_i a_j+\mu_2(D^a_i a_j+D^a_j a_i)\\
&\quad+\mu_3 c_p D^c_p c_i c_j+\mu_4(D^c_i c_j+D^c_j c_i)\\
&\quad+\mu_5 c_p D^a_p(a_i c_j+a_j c_i)+\lambda_1(A_i a_j+A_j a_i)\\
&\quad+\lambda_2(C_i c_j+C_j c_i)+\lambda_3 c_p A_p(a_i c_j+a_j c_i)\\
&\quad+\kappa_1(D^a_i c_j+D^a_j c_i+D^c_i a_j+D^c_j a_i)\\
&\quad+\kappa_2[a_p D^a_p(a_i c_j+a_j c_i)+2a_p D^c_p a_i a_j]\\
&\quad+\kappa_3[c_p D^c_p(a_i c_j+a_j c_i)+2a_p D^c_p c_i c_j]\\
&\quad+\tau_1(C_i a_j+C_j a_i)+\tau_2(A_i c_j+A_j c_i)\\
&\quad+2\tau_3 c_p A_p a_i a_j+2\tau_4 c_p A_p c_i c_j,\\
&\tilde{t}^{\mathrm{ss}}_{ij}=\lambda_1(D^a_j a_i-D^a_i a_j)+\lambda_2(D^c_j c_i-D^c_i c_j)\\
&\quad+\lambda_3 c_p D^a_p(a_i c_j-a_j c_i)+\lambda_4(A_j a_i-A_i a_j)\\
&\quad+\lambda_5(C_j c_i-C_i c_j)+\lambda_6 c_p A_p(a_i c_j-a_j c_i)\\
&\quad+\tau_1(D^a_j c_i-D^a_i c_j)+\tau_2(D^c_j a_i-D^c_i a_j)\\
&\quad+\tau_3 a_p D^a_p(a_i c_j-a_j c_i)+\tau_4 c_p D^c_p(a_i c_j-a_j c_i)\\
&\quad+\tau_5(A_j c_i-A_i c_j+C_j a_i-C_i a_j),\\
&\tilde{g}^a_i=-2(\lambda_1 D^a_i+\lambda_3 c_p D^a_p c_i+\lambda_4 A_i+\lambda_6 c_p A_p c_i\\
&\quad+\tau_2 D^c_i+\tau_3 a_p D^a_p c_i+\tau_4 c_p D^c_p c_i+\tau_5 C_i),\\
&\tilde{g}^c_i=-2(\lambda_2 D^c_i+\lambda_5 C_i+\tau_1 D^a_i+\tau_5 A_i),\\
&D^a_i=D_{ij}a_j,\qquad D^c_i=D_{ij}c_j,\\
&2D_{ij}=v_{i,j}+v_{j,i},\qquad A_i=\dot{a}_i-W_{ij}a_j,\\
&C_i=\dot{c}_i-W_{ij}c_j,\qquad 2W_{ij}=v_{i,j}-v_{j,i}.
\end{aligned}\right\} \tag{6}$$

Here $\mathbf{v}$ is the velocity, ρ is the constant density, ε_{ijk} is the alternator and a superposed dot indicates a material time derivative. The quantities p, γ, τ, κ and β are arbitrary functions of $\mathbf{x}$ and time t, and are effectively Lagrange multipliers arising from the constraints (2). When a magnetic field $\mathbf{H}$ is present the generalized body forces $\mathbf{G}^a$ and $\mathbf{G}^c$ take the forms

$$\mathbf{G}^a=\chi_{\mathrm{a}}(\mathbf{H}\ \ \mathbf{n})\mathbf{H}\cos\alpha,\qquad \mathbf{G}^c=\chi_{\mathrm{a}}(\mathbf{H}\ \ \mathbf{n})\mathbf{H}\sin\alpha, \tag{7}$$

where χ_{a} denotes the anisotropic part of the magnetic susceptibility (assumed constant). Finally H_{m} represents the energy per unit volume due to the magnetic field, and W is the bulk elastic energy per unit volume, taking the form [7, 8]

$$\begin{aligned}2W={}&K^a_1(a_{i,i})^2+K^c_1(c_{i,i})^2+K^a_2(c_i a_{i,j}c_j)^2+K^c_2 c_{i,j}c_{i,j}\\
&+K^c_3 c_{i,j}c_j c_{i,k}c_k+2K^a_3 a_{i,i}(c_j a_{j,k}c_k)\\
&+2K^c_4 c_{i,j}c_j c_{i,k}a_k+2K^{ac}_1 c_{i,i}(c_j a_{j,k}c_k)+2K^{ac}_2 a_{i,i}c_{j,j}.\end{aligned} \tag{8}$$

The theory thus provides 16 equations (2)–(5) to determine the sixteen variables a_i, c_i, v_i, β_i, p, γ, κ and τ.

3. Linear stability problems

Suppose a sample of smectic liquid crystal is confined between two fixed coaxial circular cylinders of radii r_1 and r_2 (where $r_2 > r_1$), so that the smectic layers are either circular cylinders concentric with the boundaries or are discs perpendicular to the axis of the boundaries. (Roughly speaking, these alignments correspond, respectively, to the so-called homogeneous alignment and bookshelf alignment occurring in a 'planar' cell.) We consider the effect of applying a magnetic field $\mathbf{H}$ to such an arrangement, the applied field being radial, azimuthal or axial, and also everywhere perpendicular to the initial alignments of $\mathbf{a}$ and $\mathbf{c}$. Specifically, referred to a polar coordinate system r, ϕ, z, with associated orthonormal basis $\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_z$, we shall be concerned with the following initial static states and associated applied magnetic fields:

I. $\mathbf{a}_0 = \mathbf{e}_r$, $\mathbf{c}_0 = \mathbf{e}_\phi$, $\mathbf{H} = H\mathbf{e}_z$,

II. $\mathbf{a}_0 = \mathbf{e}_r$, $\mathbf{c}_0 = \mathbf{e}_z$, $\mathbf{H} = H(r_1/r)\mathbf{e}_\phi$,

III. $\mathbf{a}_0 = \mathbf{e}_z$, $\mathbf{c}_0 = \mathbf{e}_r$, $\mathbf{H} = H(r_1/r)\mathbf{e}_\phi$,

IV. $\mathbf{a}_0 = \mathbf{e}_z$, $\mathbf{c}_0 = \mathbf{e}_\phi$, $\mathbf{H} = H(r_1/r)\mathbf{e}_r$,

H being a constant in each case (with the physical dimensions of a magnetic field). Each of these states is an equilibrium solution of the equations given above. [Of course, it may be easier experimentally to establish, for example, a radial *electric* field, rather than a radial *magnetic* field; however, since a linear stability analysis for these two arrangements results in the same mathematical problem, we refer only to the case of a magnetic field.]

We wish to determine the stability of these states to small-amplitude perturbations $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{v}}$ to $\mathbf{c}$ and $\mathbf{v}$, it being assumed that the positions of the smectic layers are unaffected (so that $\mathbf{a}$ is unchanged). We thus have

$$\mathbf{a} = \mathbf{a}_0, \qquad \mathbf{c} = \mathbf{c}_0 + \tilde{\mathbf{c}}, \qquad \mathbf{v} = \tilde{\mathbf{v}}, \tag{9}$$

with $\tilde{\mathbf{c}}$ and $\tilde{\mathbf{v}}$ small in magnitude. If strong anchoring and no-slip conditions hold on the bounding cylinders we also have

$$\tilde{\mathbf{c}} = \mathbf{0}, \qquad \tilde{\mathbf{v}} = \mathbf{0} \qquad \text{on } r = r_1 \text{ and on } r = r_2. \tag{10}$$

Consider the basic state I. For this case it is appropriate to take the perturbations $\tilde{\mathbf{c}} = \tilde{\mathbf{c}}_{\mathrm{I}}$ and $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}_{\mathrm{I}}$ to have physical components of the form

$$\tilde{\mathbf{c}}_{\mathrm{I}} = (0, 0, \theta(r))\,\mathrm{e}^{\sigma t}, \qquad \tilde{\mathbf{v}}_{\mathrm{I}} = (0, 0, v(r))\,\mathrm{e}^{\sigma t}, \tag{11}$$

it being assumed that any instability is spatially homogeneous, so that these quantities depend on the radial coordinate r but not on ϕ or z. Then with $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{v}$ as given in equation (9) the constraints (2) are satisfied identically, while the field equations (3)–(5) reduce to a pair of coupled ordinary differential equations to determine $\theta(r)$ and $v(r)$, namely

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\theta}{\mathrm{d}r} + \left(\frac{h_{\mathrm{I}} + k_{\mathrm{I}}}{r^2} - \frac{2\lambda_5\sigma}{K_2^c}\right)\theta + \frac{(\tau_5 - \tau_1)}{K_2^c}\frac{\mathrm{d}v}{\mathrm{d}r} = 0 \tag{12}$$

and

$$(\mu_0 + \mu_2 - 2\lambda_1 + \lambda_4)\left(\frac{\mathrm{d}^2 v}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}v}{\mathrm{d}r}\right) - 2\sigma(\tau_5 - \tau_1)\left(\frac{\mathrm{d}\theta}{\mathrm{d}r} + \frac{\theta}{r}\right) = 0, \tag{13}$$

where

$$h_{\mathrm{I}} = (\chi_{\mathrm{a}} H^2 r^2 \sin^2\alpha)/K_2^c,$$

$$k_{\mathrm{I}} = (K_2^c + 2K_3^c + 2K_2^a + 2K_3^a)/K_2^c. \tag{14}$$

To make progress with these equations, we will now assume† that there is an 'exchange of stabilities' at any transition, the implication being that critical values for the onset of any instability correspond to $\sigma = 0$ (thus precluding the possibility of an oscillatory instability, with σ purely imaginary). Setting $\sigma = 0$ in equation (13) leads simply to $v = 0$, while equation (12) becomes

$$r^2\frac{\mathrm{d}^2\theta}{\mathrm{d}r^2} + r\frac{\mathrm{d}\theta}{\mathrm{d}r} + b_{\mathrm{J}}\theta = 0 \tag{15}$$

for J = I, where

$$b_{\mathrm{J}} = h_{\mathrm{J}} + k_{\mathrm{J}}. \tag{16}$$

Equation (15) is to be solved subject to

$$\theta = 0 \quad \text{on } r = r_1 \text{ and on } r = r_2. \tag{17}$$

In cases II, III and IV, the analysis is somewhat similar; the corresponding perturbation fields are

$$\tilde{\mathbf{c}}_{\mathrm{II}} = \tilde{\mathbf{c}}_{\mathrm{III}} = (0, \theta(r), 0)\,\mathrm{e}^{\sigma t}, \qquad \tilde{\mathbf{c}}_{\mathrm{IV}} = (\theta(r), 0, 0)\,\mathrm{e}^{\sigma t}, \tag{18}$$

and in each case if an exchange of stabilities is assumed then $\theta(r)$ satisfies equations (15) and (17), with J = II,

† In analogous problems in nematics it can happen that the corresponding equations uncouple, and one can then *prove* that an exchange of stabilities will occur at a transition (see [9], for example). For the smectic problems considered herein such an uncoupling does not occur.

III or IV and with $b_{\rm J}$ given by (16), where

$$\left.\begin{aligned} h_{\rm II}&=(\chi_{\rm a}H^2 r_1^2\sin^2\alpha)/K_2^c,\\ h_{\rm III}&=(\chi_{\rm a}H^2 r_1^2\sin^2\alpha)/(K_2^c+K_3^c),\\ h_{\rm IV}&=(\chi_{\rm a}H^2 r_1^2\sin^2\alpha)/(K_1^c+K_2^c),\\ k_{\rm II}&=-(K_2^c+2K_3^q)/K_2^c,\\ k_{\rm III}&=(K_1^c-K_3^c)/(K_2^c+K_3^c),\\ k_{\rm IV}&=(K_3^c-2K_2^c-K_1^c)/(K_1^c+K_2^c).\end{aligned}\right\}\qquad(19)$$

We note that $h_{\rm II}$, $h_{\rm III}$, $h_{\rm IV}$ and the $k_{\rm J}$ are constants, whereas $h_{\rm I}$ depends on r (unless $H=0$).

4. Mechanical instabilities

We first consider the possibility of a so-called mechanical instability, which corresponds to the physical system finding a state of lower static energy by distorting away from an initial state even when $H=0$. In each of the cases I–IV, the differential equation governing the instability (obtained from (15) by putting $h_{\rm J}=0$) is homogeneous in r, and may readily be solved by means of the change of variable

$$r=r_1\,{\rm e}^{ls},\qquad l=\ln(r_2/r_1). \qquad (20)$$

The problem then reduces to that of solving the constant–coefficient equation

$$\frac{{\rm d}^2\theta}{{\rm d}s^2}+p_{\rm J}\theta=0,\qquad p_{\rm J}=l^2 b_{\rm J} \qquad (21)$$

for J = I, II, III or IV, subject to

$$\theta=0\quad\text{on } s=0 \text{ and on } s=1; \qquad (22)$$

here $b_{\rm J}=k_{\rm J}$, since $H=0$. It is thus found that for given values of r_1 and the material parameters, there is a threshold outer radius $r_2=r_{\rm c}$ given by $p_{\rm J}=\pi^2$ beyond which a mechanical instability will occur. Explicitly, the threshold radii for the four cases are given by

$$r_{\rm c}=r_1\exp[\pi/(k_{\rm J})^{1/2}], \qquad (23)$$

for J = I, II, III or IV, provided, of course, that $k_{\rm J}$ is positive.†

5. Fréedericksz transitions

We now turn our attention to the occurrence of Fréedericksz transitions induced by the application of magnetic fields. Of course, to ensure that the initial static states I–IV are realizable before the field is applied (i.e. with $H=0$) it is necessary to choose $r_2<r_{\rm c}$; then the application of an incrementally increasing H should lead to a transition. In cases II–IV, the differential equation (15) reduces under (20) to (21), the constant $p_{\rm J}$ now involving a contribution from H. It is found that the threshold fields $H_{\rm c}$ are given by

$$b_{\rm J}=(\pi/l)^2 \qquad (24)$$

for J = II, III, IV, with $b_{\rm J}$ as in equations (16) and (19). The result for case II agrees with that of Atkin and Stewart [6], obtained via consideration of static energies (and given in a different notation in [6]).

Case I is slightly different in that, although equation (15) for J = I reduces under (20) to an equation of the form (21), the coefficient $p_{\rm I}$ is not a constant, but depends on s (for $H\neq 0$)—so the solution is not as for the other cases. Writing equation (15) in the form

$$\frac{{\rm d}^2\theta}{{\rm d}r^2}+\frac{1}{r}\frac{{\rm d}\theta}{{\rm d}r}+\left(k_{\rm I}+h\frac{r^2}{d^2}\right)\frac{\theta}{r^2}=0, \qquad (25)$$

where

$$d=r_2-r_1,\qquad h=(\chi_{\rm a}H^2 d^2\sin^2\alpha)/K_2^c, \qquad (26)$$

we proceed in two complementary ways. First we obtain results based on a 'narrow gap' approximation, and secondly we consider an exact solution for the case $k_{\rm I}\leqslant 0$.

With a new variable x and a parameter δ defined by

$$r=r_1(1+\delta x),\qquad \delta=d/r_1, \qquad (27)$$

the problem becomes that of solving

$$\frac{{\rm d}^2\theta}{{\rm d}x^2}+\frac{\delta}{1+\delta x}\frac{{\rm d}\theta}{{\rm d}x}+\left[\frac{k_{\rm I}\delta^2}{(1+\delta x)^2}+h\right]\theta=0 \qquad (28)$$

subject to the conditions

$$\theta=0\quad\text{on } x=0 \text{ and on } x=1. \qquad (29)$$

For a narrow gap we take $\delta\ll 1$ and seek a solution of the form

$$\begin{aligned}\theta&=\theta_0(x)+\delta\theta_1(x)+\delta^2\theta_2(x)+\dots,\\ h&=h_0+\delta h_1+\delta^2 h_2+\dots.\end{aligned} \qquad (30)$$

Substituting equation (30) into (28) and equating corresponding powers of δ we obtain a sequence of problems of the form

$$\frac{{\rm d}^2\theta_i}{{\rm d}x^2}+h_0\theta_i=R_i\qquad (i=0,1,2,\dots). \qquad (31)$$

with

$$\theta_i=0\quad\text{on } x=0 \text{ and on } x=1, \qquad (32)$$

†Although the nine elastic constants in W are known to satisfy certain inequalities, the signs of the $k_{\rm J}$ are unknown at present; if $k_{\rm J}<0$ then there can be no mechanical instability.

where

$$R_0 = 0, \qquad R_1 = -h_1\theta_0 - \frac{d\theta_0}{dx},$$

$$R_2 = -h_1\theta_1 - h_2\theta_0 - \frac{d\theta_1}{dx} + x\frac{d\theta_0}{dx} - k_I\theta_0,$$

$$R_3 = -h_1\theta_2 - h_2\theta_1 - h_3\theta_0 - \frac{d\theta_2}{dx} + x\frac{d\theta_1}{dx} - x^2\frac{d\theta_0}{dx}$$

$$-k_I\theta_1 + 2k_I x\theta_0, \ldots .$$

A straightforward integration of these equations yields the threshold field

$$h = \pi^2 - (k_I + \tfrac{1}{4})\delta^2 + (k_I + \tfrac{1}{4})\delta^3 + \ldots . \qquad (33)$$

In the limit of a very narrow gap ($\delta \to 0$), case I becomes equivalent to the problem of a Fréedericksz transition in a smectic sample in 'homogeneous alignment' held between parallel plates, when $\mathbf{a}$ is normal to the plates, $\mathbf{c}_0$ is parallel to the plates, and the applied field is parallel to the plates and normal to $\mathbf{c}_0$. For this latter problem the threshold field is given by $h = \pi^2$; equation (33) shows that the correction to this (associated with the curvature of the annular geometry) comes in only at $O(\delta^2)$, and involves only the elastic constants in k_I.

Lastly we consider case I when $k_I \leqslant 0$. Equation (25) has the general solution

$$\theta(r) = AJ_q(h^{1/2}r/d) + BY_q(h^{1/2}r/d), \qquad (34)$$

where $q = (-k_I)^{1/2}$, and J_q and Y_q denote Bessel functions. Application of the boundary conditions (17) yields the relationship

$$J_q(h^{1/2}r_1/d)Y_q(h^{1/2}r_2/d) - J_q(h^{1/2}r_2/d)Y_q(h^{1/2}r_1/d) = 0 \qquad (35)$$

to determine h. This is rather cumbersome, but in the special event that $k_I = -\frac{1}{4}$ the solution (34) simplifies to

$$\theta = \frac{1}{r^{1/2}}\left\{A_1 \cos\left[h^{1/2}\left(\frac{r - r_1}{r_2 - r_1}\right)\right] + B_1 \sin\left[h^{1/2}\left(\frac{r - r_1}{r_2 - r_1}\right)\right]\right\}. \qquad (36)$$

The boundary conditions (17) then give $A_1 = 0$, and it is found that the threshold field h is given by $h = \pi^2$.

6. Summary and discussion

We have presented a catalogue of the threshold magnetic fields H_c required to induce a Fréedericksz transition in a smectic sample confined to a cylindrical annulus. Various configurations of initial director pattern and of the applied field have been studied. The threshold fields are given by equations (24), (16) and (19) for cases II, III and IV, and by equations (33) or (35) with (14) and (26) for case I. The H_c depend on five distinct combinations of the nine elastic constants, so observation of critical phenomena in the experimental arrangements considered here should provide a means of measuring these combinations, and hence of deducing values of K_1^c, K_2^c, K_3^c, K_2^a and K_3^a. For cases where a mechanical instability is possible, it is necessary in an experiment to ensure that the outer radius r_2 does not exceed the appropriate critical radius r_c. Additionally, we note that the existence of a threshold r_c means that a Fréedericksz transition may be induced with a relatively small field H: if the system is already 'near' a mechanical instability then only a small field will be needed to 'push' it into a transition. On the other hand, an approximation to the set-up for a purely mechanical instability may be achievable by means of an 'almost-cylindrical' cone in place of the outer cylinder; then the 'effective' radius will vary slowly along the axis of the system and at the point where it takes the critical value, a change of orientation pattern should be observable. Measurements of $r_2 = r_c$ at this point will provide an estimate of k_J via equation (23).

When the applied field is azimuthal, order-of-magnitude arguments for nematics (see [9, 5]) indicate that typically a line current of about 20 A would be needed to generate the critical field. If the elastic constants for smectics are of the same order of magnitude as those for nematics then the above analysis shows that a comparable value would be required for the smectic case; with a radial field, a similar critical value would be needed. In the case of an axial field Strigazzi [10] estimates that for a nematic 'cell' with $r_1 = 1$ mm and $d = 20\,\mu$m, the critical field H_c will typically be about 2×10^5 A m^{-1}; one might expect a similar field for a smectic sample in such a cell. For larger values of d/r_1 this critical value will be smaller.

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